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Impedance Matrix and its Use for Modeling Axially Polarized Piezoceramic Cylindrical Resonators

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Abstract - Analytical formulas for the elements of the impedance matrix $\mathbf{Z}$ of a cylindrical piezoelectric resonator vibrating in a shear mode are established. The input impedance was calculated employed the established elements of the impedance matrix $\mathbf{Z}$. The analysis presented in this paper can be utilized for the design and construction of the cylindrical piezoelectric sensors, transducers and multi-layer resonators.

I. INTRODUCTION

The impedance matrix is a crucial element in modeling the electroacoustic behavior of piezoelectric resonators, filters, sensors and transducers [1,2].

In this study, using a continuum electromechanical model the impedance matrix has been established for a piezoceramic cylindrical resonator vibrating in a shear mode. Polarization is along the longitudinal z axis. Inner and outer cylindrical surfaces can be loaded with an arbitrary mechanical impedance (e.g., viscoelastic liquid).

Analytical formulas obtained for the impedance matrix enabled the determination of the input impedance (admittance) of the piezoceramic cylindrical resonator loaded with a viscoelastic liquid on the outer or inner cylindrical surface. Elements of the impedance matrix are combinations of the Bessel and Neumann functions and depend on the frequency, material and geometrical parameters of the cylindrical resonator. In the analysis, mechanical losses in a piezoelectric ceramic were taken into account.

The analysis presented in this study also describes the case of multi-layer piezoelectric and non-piezoelectric cylindrical structures. This can be useful in the design of multi-layer cylindrical sensors composed of one piezoelectric layer and a stack of mechanical cylinders fabricated from various metals.

In this paper the authors consider shear vibrations of an infinitely long piezoceramic cylinder, see Fig.1. The electrodes are deposited on the inner and outer cylindrical surface. For such a manner of excitation, the only non-zero component

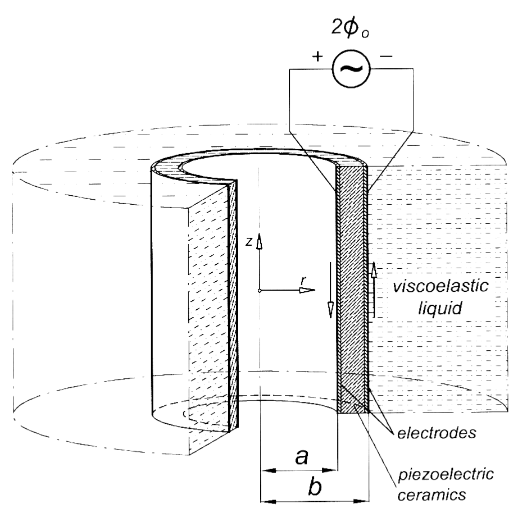

Fig.1. Infinite (along the z axis) piezoceramic cylinder with the outer surface ($r = b$) loaded by a viscousoelastic liquid. The arrows indicate the shear vibrations of the cylinder. Polarization is along the longitudinal z axis of the cylinder. Electrodes are deposited on the outer ($r = b$) and inner ($r = a$) cylindrical surfaces.

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of vibrations is the shear displacement u_z along the vertical z axis.

II. THEORY

A. Piezoelectric Constitutive Relations

To describe the electro-elastic behavior of a piezoceramic cylinder (see Fig.1) the mechanics of continuum and the theory of linear piezoelectricity were used. The constitutive relations were chosen as follows:

$$T_{rz} = c_{44}^{E} S_{rz} - e_{15} E_r \qquad (1)$$

$$D_r = \varepsilon_{11}^{S} E_r + e_{15} S_{rz} \qquad (2)$$

where: T_{rz}, D_r, S_{rz}, E_r are the components of stress, electric displacement, strain and electric field respectively; c_{44}^{E}, e_{15} and ε_{11} are the elastic modulus, piezoelectric coefficient and dielectric constant of the piezoelectric ceramics.

B. Electro-elastic Equations of Motion

The governing equations of motion in cylindrical coordinates r, θ, z are independent of angle θ and vertical coordinate z.

$$c_{44}^{E}\left(u_{z,rr} + \frac{1}{r} u_{z,r}\right) + e_{15}\left(\Phi_{,rr} + \frac{1}{r}\Phi_{,r}\right) = \rho \ddot{u}_z \qquad (3)$$

$$e_{15}\left(u_{z,rr} + \frac{1}{r} u_{z,r}\right) - \varepsilon_{11}^{S}\left(\Phi_{,rr} + \frac{1}{r}\Phi_{,r}\right) = 0 \qquad (4)$$

where: u_z, Φ and ρ are the mechanical displacement, electrical potential and density of the piezoelectric ceramics.

We will analyze the problem assuming harmonic dependence on time: $\exp(j\omega t)$, $j = (-1)^{1/2}$.
Eliminating Φ from Eqs.3 and 4 the following equation for u_z is obtained [3]:

$$u_{z,rr} + \frac{1}{r} u_{z,r} + \lambda^2 u_z = 0 \qquad (5)$$

where: $\lambda = \omega\left(\rho / c_{44}^{D}\right)^{1/2}$ and ω is an angular frequency.

The general solution of Eq.5 is [3]

$$u_z = A_1 \cdot J_0(\lambda r) + A_2 \cdot Y_0(\lambda r) \qquad (6)$$

where: J_0 and Y_0 are the zero-order Bessel and Neumann functions, A_1 and A_2 are arbitrary constants.

III. PIEZOELECTRIC IMPEDANCE MATRIX

A piezoelectric cylinder is represented as 3-port network composed of one electrical port and two acoustic ports. Mechanical forces F_1 and F_2 at the inner and outer surface and electric voltage V_3 across the electrical port were chosen as dependent variables.

$$\begin{bmatrix} F_1 \\ F_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ I_3 \end{bmatrix} \qquad (7a)$$

The electric field E_r is treated as quasi-static. To obtain the elements of the matrix Z the following expressions to the mechanical forces F_1 and F_2 were used:

$$F_1 = -2\pi a \cdot T_{rz}(\lambda a) \qquad (7b)$$

$$F_2 = -2\pi b \cdot T_{rz}(\lambda b) \qquad (7c)$$

where: $T_{rz} = c_{44}^{D} u_{z,r} - h_{15} D_r$

Moreover, the expression for electrical displacement D_r was integrated over the variable r (radius) from $r = a$ to $r = b$. Finally, after some algebra, we arrived at the following formulas for the elements of the impedance matrix Z:

$$z_{11}=2\pi a\frac{1}{j}(\rho c_{44})^{1/2}\frac{(J_0(\lambda b)\cdot Y_1(\lambda a)-Y_0(\lambda b)\cdot J_1(\lambda a))}{(J_0(\lambda b)\cdot Y_0(\lambda a)-J_0(\lambda a)\cdot Y_0(\lambda b))}$$

$$z_{22}=2\pi b\frac{1}{j}(\rho c_{44})^{1/2}\frac{(J_0(\lambda a)\cdot Y_1(\lambda b)-Y_0(\lambda a)\cdot J_1(\lambda b))}{(J_0(\lambda b)\cdot Y_0(\lambda a)-J_0(\lambda a)\cdot Y_0(\lambda b))}$$

$$z_{12}=z_{21}=-\frac{4}{j\omega}c_{44}^{D}\frac{1}{(J_0(\lambda b)\cdot Y_0(\lambda a)-J_0(\lambda a)\cdot Y_0(\lambda b))}$$

$$z_{13}=z_{31}=z_{23}=z_{32}=\frac{h_{15}}{j\omega},\ z_{33}=\frac{1}{j\omega C_0};\tag{7d}$$

where: $C_0=2\pi\varepsilon_{11}^{S}\frac{1}{\ln(R_2/R_1)}$ and $h_{15}=\frac{e_{15}}{\varepsilon_{11}^{S}}$

It should be noted that, the impedance matrix Z is symmetrical $z_{12}=z_{21}$, and $z_{11}\neq z_{22}$.

IV. ELECTRICAL INPUT IMPEDANCE (ADMITTANCE)

The impedance matrix method enables the determination of the input impedance $\mathbf{Z_{IN}}$ of a cylindrical transducer. For the case of a free piezoelectric cylinder we have

$$Z_{IN}=z_{33}+\frac{z_{31}(z_{13}z_{22}-z_{23}z_{12})+z_{32}(z_{23}z_{11}-z_{13}z_{21})}{(z_{21}z_{12}-z_{11}z_{22})}\tag{8}$$

The formula 8 for the input impedance is also valid when piezoelectric cylinder is loaded with the acoustic impedance Z_c on the outer cylindrical surface: In this case the element z_{22} in Eq.8 should be replaced with $z_{22}+2\pi b Z_c$.

V. NON-PIEZOELECTRIC CYLINDRICAL RESONATORS

The impedance matrix representing a non-piezoelectric (mechanical) cylinder is a special case of Eq.7a (upper left 2x2 sub-matrix).Equivalent circuit corresponding to this sub-matrix Z is presented in Fig.2.

Note that: $(z_{11}-z_{12})\neq(z_{22}-z_{12})$

VI. COMPOUND RESONATORS MULTI-LAYER RESONATORS

$$\begin{bmatrix}F_1\\ v_1\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}F_2\\ v_2\end{bmatrix}\tag{9}$$

$$[A]=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\frac{1}{z_{12}}\begin{bmatrix}z_{11} & (z_{11}z_{22}-z_{12}^2)\\ 1 & z_{22}\end{bmatrix}\tag{10}$$

$$[A]=[A_1]\times[A_2]\times\ldots\times[A_N]\tag{11}$$

The impedance matrix method can be applied for analyzing multi-layer piezoelectric cylindrical transducers composed of one active (piezoelectric) layer and a stack of passive (mechanical) layers (e.g., metallic). In this case to describe the problem it is more convenient to use the chain matrix A.

The behavior of the mechanical (passive) stack of N cylinders is described with the chain matrix $\mathbf{A}$ (Eq.11) which is a product of N chain matrices $A_1, A_2, ..., A_N$ of the subsequent cylindrical layers. The elements of the chain matrix $\mathbf{A}$ are related to the elements of matrix $\mathbf{Z}$, see Eq.10.

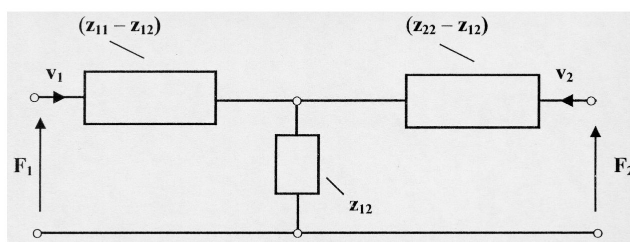

Fig.2. Equivalent circuit corresponding to the impedance matrix Z of a passive (mechanical) cylinder.

VII. CONCLUSIONS

We have established the analytical formulas for the impedance Z matrix of a long piezoelectric cylinder vibrating in a shear mode.

This analytical model can be applied in the investigations of the electroacoustic behavior of the following devices: a) sensors (viscosity and elasticity of liquids), biosensors, chemical sensors b) accelerometers c) filters d) multi-layer cylindrical resonators and transducers

Immitance diagrams calculated by using the impedance Z matrix method (Eq.8) conform to those evaluated from the electromechanical model (Eq.18 from [4]). This can prove the correctness of the analytical formulas for the elements of the impedance matrix Z established in this paper.

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